Social Balance and Signed Network Formation Games

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ABSTRACT

This paper presents a game-theoretic approach that models the formation of signed networks which contain both *friendly* and *antagonistic* relations. In our model, nodes have a *pleasure* through their links to the others which is defined according to their triadic relations. They have a self centered goal based on the balance theory of signed networks: *each node tries individually to minimize its stress from undesirable relations with other nodes by maximizing the number of balanced triangles minus the number of unbalanced triangles she participates in.* Since nodes act selfishly and don't consider the social welfare, many theoretical problems about existence of the equilibrium, convergence and players' interaction are raised and verified in this paper. We prove the NP-hardness of computing best-response and give an approximation for it by using quadratic programming and rounding its solution. We show that there is a tight relation between players' best-responses. This result leads to a proof for convergence of the game. In addition, we report some experimental result on three online datasets. We show that as nodes play their best-response and time goes forward, the total pleasure of the network increases monotonically. Finally, we introduce a *smartness factor* for social media sites that helps people to measure the amount of deviation from best possible strategy and suggest a new model that is more adapted to these media. This measure can also be applied in the verification of all kinds of network formation games.

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1. INTRODUCTION

Nowadays, one of the most important technologies for managing social relations are online social networks. Over the past years, analysis of the interaction between people in online social media has been taken into consideration. There exist a wide variety of relations on the online social networks; most of such relations have positive aspect, such as friendship, like, trust, and follow. Alongside these fine relations, there are always some negative relations such as antagonism, distrust, and unlike. The main focus of social network researches has been on the positive relations; however, some recent works investigate the interplay between positive and negative relations in social media sites [15, 8].

Social network analysis is usually based on the network theory concepts studied on the underlying network (graph). Such a network consists of a set of nodes along with a set of edges that connect them.

Our focus in this work is on *signed networks*. In signed networks, every edge has a sign which takes two values: positive ($+$) for positive relations or negative ($-$) for negative relations.

One of the most important problems in the signed networks is the interplay between signs and their effects on the

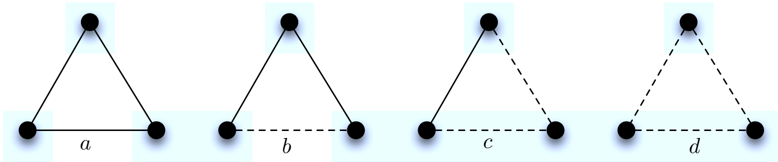

Figure 1: Possible undirected triad in a signed network: triad (a) and (c) are balanced and relatively stable, but triad (b) and (d) are unbalanced and susceptible to break apart. Full and dashed lines represent positive and negative relations respectively.

dynamics of the network. Structural balance theory is a basic framework to perceive such interactions between nodes' relations. In this work we present a robust evolutionary model that allows us to see the power of structural balance theory in signed networks.

Balance theory was formed by Heider [11] in 1946. Harary [10] developed some notion about this theory and, later on, Cartwright and Harary [4] extended the results to a graph-based concept. Structural balance theory describes attitudes of individuals to reduce cognitive dissonance among each other. When people set up dyadic relations that contain both positive and negative interactions, four different types of triad relations would be created (Figure 1). In conformity to this theory, we can classify these triangles in 2 classes: *balanced* and *unbalanced*.

A balanced triangle shows a relationship between three people without cognitive disturbance. Such as Figure 1-a that shows three people who are mutual friends and Figure 1-c shows two friends who have a mutual enemy. But in an unbalanced triangle we observe a psychological stress into the relationship. In the Figure 1-b we see a pair of enemies that have a common friend. In this situation these enemies feel pressure from their common friend to become friends; or else there is a pressure from one of the enemies on its friend against the other. In Figure 1-d we watch a challenging state. When three people are enemies, there is a chance for two of them to join together against the third person.

A usual variant of the structural balance theory is weak structural balance proposed by Davis [6]. This variant of the theory relaxes the assumption that "the enemy of my enemy is my friend" and considers a triangle like the one in Figure 1-d as a balanced triangle. Leskovec et al. [15] considered the online social media and counted the number of each triangle in corresponding datasets. They report that the number of triangles with three negative links appear much more than in the randomly generated graphs. These results confirm the validity of the weak balance theory of Davis in the online social media.

Many types of networks connect different individuals and the formation of such networks depends on the decisions of many participants. Network formation models have been widely used in order to characterize how network structures form and how they are affected by individuals' decisions [13]. Aumann and Myerson [2] presented the first network formation model describing the formation of a network in the context of cooperative games with communication structures. Later, several models in this context presented in both ontime and dynamic perspective [19, 12, 3, 5, 14, 9, 7]. Recently Arount van de Rijt [20] attempted to investigate the structure of signed graphs by this type of modeling. He presented a best-response model which has a node utility function that takes its maximum when the number of balanced triads that involve the node are maximized. He proved couple of statements about balanced and stochastically stable graphs that appear in the model. The main difference of this work from the aforementioned works is that he studied the evolution of sign changes while leaving the network structure fixed.

What we present here is a network formation game for studying the evolution of signed networks based on the structural balance theory. The players are nodes and their strategy is creating positive/negative/neutral edges to all other nodes. The edges are undirected and, once created, represent bilateral friendship/antagonism relationship among players, regardless of which node created it. The sum of these edges is the resulting signed graph. Each node has a self-centred goal: to minimize the stress of undesirable relations between itself and all other nodes. We formulate this goal in a best-response model by using the balance theory: each node attempts to maximize the number of balanced triangles minus the number of unbalaced ones. We show that computing the best-response in this model is NP-Hard and present a convex optimization method for its approximation. We also analyse the Nash equilibrium of the model and provide some relevant results.

1.1 Related works

There are many works considering structural balance theory in signed networks. Antal et al. [1] presented simple dynamical rule for resolving unbalanced triads and investigated the resulting evolution of the signed networks. They study the conditions and times under which their model reaches a balanced state. They also characterize a class of jammed states; states in which unbalanced triangles exist and the change of any link's sign increases the number of unbalanced triangles. They proved that jammed states turn out to be much more numerous than balanced states. Following Antal's work, Marvel et al. [17] define an energy landscape for signed networks and find out that the jammed states with higher energy are not only very rare, but they also have inherently greater structural complexity, as measured by their number of balanced cliques.

Ludwig and Abell [16] proposed a model that used balance theory to describe the development of social networks. Their model has two parameter, the first is a friendliness index that help to determine the probability of a link's sign to be positive. The second is a uniformly distributed threshold that indicates the quantity of unbalanced triangles that each node tolerates.

Some works such as [15, 18] examine the validity of structural balance in online social media sites. Leskovec et al. show that aspects of balance theory hold more strongly on the links that are reciprocated (consisting of directed links in both directions between two nodes). Szell shows that a vast majority of changes in the signed networks are due to the creation of new positive and negative links, not due to switching of existing link from plus to minus or vice versa.

1.2 The Best-Response Model

We consider a repeated game with n players labeled by $1, 2, \cdots, n$. Each player i chooses a $(n-1)$-dimensional vector $L_i \in \{-1, 0, +1\}^{n-1}$ with components l_{ij} where $l_{ij} \in \{-1, 0, +1\}$ for each $j \in \{1, \cdots, N\} \setminus \{i\}$ as its strategy toward the others. At time t, by combining players' strategies $L = (L_1, L_2, , L_n)$ an undirected signed graph G_t with vertices $\{1, \cdots, n\}$ is formed. There is an edge between vertices i and j in G_t iff $l_{ij} \neq 0$. The sign of this edge is equal to the sign of l_{ij}. The pleasure received by player i under L is defined to be

$$p_i(L) = (\triangle_{b_i} - \bigtriangledown_{ub_i}) \tag{1}$$

where $\triangle_{b_i}$ is the number of balanced triangles that i participated in and $\bigtriangledown_{ub_i}$ is the number of unbalanced triangles. By choosing the best possible strategy, players attempt to maximize their pleasure in the network (best-response of player i). Player i is just permitted to play its best response at times $t = kn + i$ for $0 \leq k$. We say that a strategy profile L is a (pure) Nash equilibrium if for each player i, and for all L' that differ from L only in the i^{th} component, $p_i(L) \geq p_i(L')$. In our model computing the best-response for each player is not possible in polynomial time. A proof for this is given in section 2. So we should propose a method for its approximation. We will prove that the best response is an integral answer of a famous semidefinite programming.

2. GAME INTRACTABILITY

In this section, we provide a theoretic interpretation of the game. Let us start with introducing some notations and an easy yet supportive lemma:

- v-triangle: Is a triangle whose vertex set includes v.
- uv-triangle: Is a triangle whose vertex set includes both u and v.
- Δabc: Is a triangle whose vertex set is $\{a, b, c\}$.
- $\delta(A, B)$: The number of triangles whose vertex set include all vertices in A and at least one vertex from B.

LEMMA 2.1. *Assume u is playing its best response in G_t. Adding any set of edges to u with any sign does not decrease u's pleasure.*

PROOF. Let v be a vertex that u is not connected to. Connect u to v with positive sign. Let a be the number of balance uv-triangles and b be the number of unbalanced ones. So, the pleasure of u changes by $a - b$. If we change the sign of uv then the pleasure of u changes by $b - a$ (every balanced uv-triangles becomes unbalanced and vice versa). So, we must have $a = b$ or u is not playing its best response. We use the same argument for every other vertex that u is not connected to and obtain a graph in which u is connected to every other vertex. A side corollary of this result is that there exist a best response playing for player u that connects it to all other vertices. □

One important question while we are studying a game is the computational status of various solution concepts such as best response and Nash equilibirum. We prove that in our game, computing best-response for each player at each time is NP-hard.

THEOREM 2.2. *Given a strategy profile $L \in L_0 \times L_1 \times \cdots \times L_n$ and player $i \in \{1, .., N\}$, computing the best response of i is NP-hard.*

PROOF. The proof follows by reducing from the *Max-Cut* problem. In the *Max-Cut* problem, we are interested in partitioning the vertices of a given graph G into two sets A and B so as to maximize the number of edges between A and B. Let G be a graph that we want to find its maximum cut. Put a negative sign for every edge of the graph. Add a new vertex v to G. We prove that v's best response is, in fact, equivalent to finding a maximum cut in G. By Lemma 2.1, we find a best response playing by v in which v draws edge to every vertex of G. Now partition G's vertices into two subsets:

- $A = \{u | l(vu) = -1\}$
- $B = \{u | l(vu) = +1\}$

We prove that (A, B) is a maximum cut. We have:

$$\begin{aligned} BR(v) = \triangle_{b_v} - \bigtriangledown_{ub_v} &= E(A, B) - E(A, A) - E(B, B) \\ &= 2E(A, B) - E(G) \end{aligned}$$

where $E(X, Y)$ is the number of edges with one endpoint in X and one in Y and $E(G)$ is the number of edges in G. Since v is playing best response, $E(A, B)$ must be maximum over all possible cuts. □

If we allow the edge signs to be real values in the interval $[-1, 1]$ instead of integer values then we can compute the best response of every player in by quadratic programming. Assume we want to compute the best response of u. We first remove u from the network and model its best response by the following QCQP optimization problem.

$$\begin{aligned} &minimize - X_i A X_i^T \\ &subject \ \ to - 1 \leq x_{ij} \leq +1, \quad j = 1, \cdots, n-1 \end{aligned} \tag{2}$$

where n is the number of nodes in the network, the $(n-1) \times (n-1)$ matrix A is the weighted adjacency matrix of $G - \{i\}$, and x is the best-response of node i to the remainder of the network (Figure 2). We know that a triangle containing an even number of negative edges is defined as balanced while a triangle with an odd number of negative edges is unbalanced. Therefore, any integer solution to the above formulation computes the number of balanced triangle minus the number of unbalanced triangles that contain u.

We next consider the repeated game (in which one vertex is picked at each time and plays its best response). Our focus is the convergence of this game. First, we start with the following lemma:

LEMMA 2.3. *Let u and v be two vertices that play their best responses at times t and $t + 1$, respectively. Then we have*

- *If $l_{vu} = +1$, then we have: $l_{uk} = l_{vk}$, for all $k \neq u, v$ with $l_{uk} \neq 0$.*

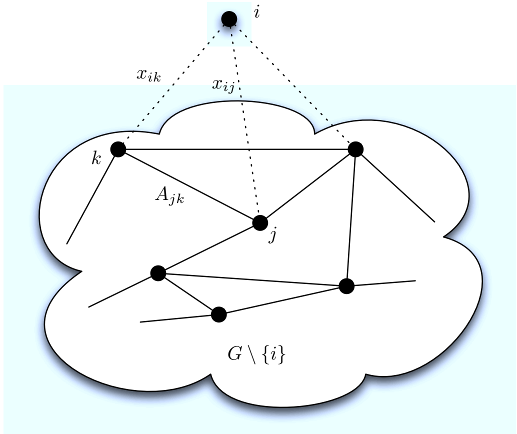

Figure 2: Node i's best-response computation. When node i decides to change her current strategy to achieve a better pleasure, she ignores her current relations and computes her best-response by QCQP optimization. x_{ik} and x_{ij} represent unknown relations that node i attempts to find according to the relations between nodes k and j (i.e. A_{jk}).

- *If $l_{vu} = -1$, then we have: $l_{uk} = -l_{vk}$, for all $k \neq u, v$ with $l_{uk} \neq 0$.*

Proof. W.l.o.g suppose that at time t, we have $l_{uv} = +1$. The case $l_{uv} = -1$ is treated similarly. We use u's best-response to build a strategy for v and prove that this strategy is strictly better than all other strategies for her. Define the following sets of vertices:

- X: The set of vertices which are connected to vertex u at time t.
- Y: The set of vertices which are connected to vertex v and have no edge to vertex u at time t.

Define $f(u)$ to be the pleasure of vertex u in graph $G_t[X \cup \{u\}]$ (the induced subgraph of G_t over the vertices of $X \cup \{u\}$). Let L_v be a strategy for v in which $l_{vk} = l_{uk}$ for each $k \in X$. At this time, the pleasure of v is equal to $f(u) + |X|$, because its edges to X have the same sign as u's edges to X and for every $k \in X$ we have one balanced uv-triangle, Δuvk. Next, at each step we choose a vertex k' from Y and set $l_{vk'}$ in such a way that it strictly increases the pleasure of v. We will prove that it is always possible.

At time t, there isn't any edge between k' and u. By Lemma 2.1 adding an edge between k' and u does not change u's pleasure. So the number of balanced uk'-triangles and unbalanced uk'-triangles are equal (If the edge uk' exists). This means that the number of uk'-triangles is even. This is true for every other $y \in Y$. Consider two of the vertices in Y, such as y_1 and y_2. We claim that there is no edge between these two vertices. Suppose that y_1y_2 is an edge of G_t. Again by using Lemma 2.1 we can show that the number of uy_i-triangles for $i = 1, 2$ is even. The contradiction happens here:

$$\delta(\{u\}, \{y_1, y_2\}) = \delta(\{u\}, \{y_1\}) + \delta(\{u\}, \{y_2\}) - \delta(\{u, y_1, y_2\}, \emptyset)$$

and we have $\delta(\{u, y_1, y_2\}, \emptyset) = 1$. This contradicts our assumption because left hand side of the above equality is even while the right hand side is odd. So there is no edge between any $y_1, y_2 \in Y$ and $G[Y]$ is independent set.

Now, consider vertices v and k'. The number of uk'-triangles in G_t by assuming existence of uk' is even (note that in this graph vk' is available and there is no edge between vertices of Y). So, the number of vk'-triangles in the graph G_{t+1} and in the absence of uk' is odd because $\Delta uvk'$ won't be enumerated. So we can choose a suitable sign for the edge vk' which makes the number of balanced vk'-triangles more than the number of unbalanced ones. The proof for the claim is now complete and the pleasure of vertex v is at least $f(x) + |X| + |Y|$ which means $BR(v) \geq f(x) + |X| + |Y|$.

Now, suppose that v chooses another strategy S. By using Lemma 2.1 we can assume that in S there is an edge from v to every vertex in X. Define X_1 to be the set of vertices which have edges to both u and v and their edges to these vertices have the same sign (In graph G_{t+1} constructed by adding S to G_t). Similarly define X_2 to be the set of vertices which have edges to both of these vertices but with different signs. Clearly we have, $X = X_1 \cup X_2$. Define X'_1 and X'_2 in a manner similar to X_1 and X_2 over the graph G_t. We have:

$$BR(v) = f(v) + |X_1| - |X_2|$$

$$BR(u) = f(u) + |X'_1| - |X'_2|$$

We proved that $BR(v) \geq f(u) + |X| + |Y|$, hence by extending this inequality we obtain:

$$f(v) + |X_1| - |X_2| \geq |X| + |Y| + f(u) \quad (3)$$

Now suppose that at time t vertex u uses the best response of vertex v at time $t+1$. Therefore:

$$BR(u) \geq f(v) + |X'_1| - |X'_2| - |X_2| - |Y|$$

From $BR(u) = f(u) + |X'_1| - |X'_2|$, thus:

$$f(u) \geq f(v) - |X_2| - |Y| \quad (4)$$

From Inequalities 3 and 4 we have:

$$|X_1| \geq |X| \Rightarrow |X_2| \leq 0$$

And the proof is complete. □

Lemma 2.3 reveals important information regarding the structure of the game equilibrium and its convergence time. We show that the equilibria of this repeated game are complete graphs and convergence happens after $O(n)$ steps.

Theorem 2.4. *After one round the graph of the game will be complete.*

Proof. Suppose that there is no edge between two vertices u and v, and u plays its best response first. Assume that in v's turn, v does not draw an edge to u. So by using Lemma 2.1 we know that drawing an edge with positive sign between these two vertices will not change v's pleasure. This is also true for an edge with negative sign. Define X to be the set of u's neighbors after playing its best response. Now consider two different signing patters for the edges from v to X:

Table 1: A method for making a signed graph undirected. This works because our datasets have very small fraction of reciprocated edges and in the balance theory it is reasonable to expect that the reciprocated edges have same signs. $'+'$, $'-'$, and $'\phi'$ represent positive,negative, and unknown relations between nodes u and v respectively.

k	$u \rightarrow v$	$u \leftarrow v$	$u \leftrightarrow v(undirected)$
0	$+$	$+$	$+$
1	$+$	ϕ	$+$
2	ϕ	$+$	$+$
3	$-$	$-$	$-$
4	$-$	ϕ	$-$
5	ϕ	$-$	$-$
6	$+$	$-$	ϕ
7	$-$	$+$	ϕ

- E_1: For each $x \in X$, vx's sign is as same as the sign of ux.
- E_2: For each $x \in X$, vx's sign is invert of the sign of ux.

When we consider a positive edge between u and v, from Lemma 2.3 we know that E_1 signing pattern is strictly better than all other strategies. However, when we consider a negative edge between these vertices, E_2 is the best strategy for v. But as we justified earlier, these two strategies must have the same pleasure for v, but this implies a contradiction. □

THEOREM 2.5. *After one round ($O(n)$ steps) the game graph converges to an equilibrium.*

PROOF. Suppose that we are at $(n+1)$'th step and it is player one's turn to play its best response. W.l.o.g suppose that the edge between player 1 and player n has positive sign. We know that at the previous turn player n has played its best response. From the proof of the Theorem 2.4 we know that n has drawn edges to all vertices of the graph and its best response follows the properties mentioned in Lemma 2.3, i.e., all the (n, i) edges have the same sign as $(1, i)$ edges have. So this lemma also holds for player 1 and its current edges form its best response. So it has no desire for changing its outgoing edges. □

3. EXPERIMENTAL RESULTS

In this section we explore the above theoretical results experimentally on realistic datasets. For this, we consider a simulation scenario in which we start with an initial network and then in each step nodes iteratively change their strategy. Our simulations are conducted on both random networks and online social media datasets. The results are discussed in the following sections.

As mentioned in the previous section, computing best response in each step is NP-hard (Theorem 2.2) and we need a method for its approximation. As we said, the relaxed problem in which the integral constraints are removed, can be solved by a QCQP (Equation 2). We solve this QCQP and round its solution (values near $\zeta \in \{-1, 0, +1\}$ are rounded to ζ). This would give a good approximation for the problem. Even though we round the obtained solution to the nearest integer value, this process has little effect on the accuracy of the optimized values achieved by relaxation vs. the truly optimal values.

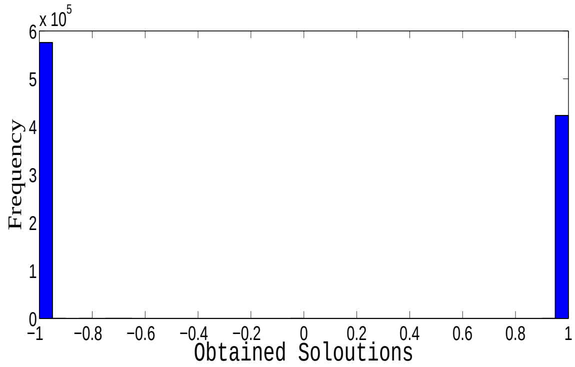

Figure 3: QCQP solution histogram. Frequency of the solutions in a subnetwork of Wikipedia dataset with 1000 nodes. Only very few of the solutions needs rounding and this result shows that our QCQP maximization method has a negligible noise.

Our experiments on some real subnetworks as well as random graphs show that almost all solutions to the QCQP problem are very close to $+1$ or -1, and there are very few different solutions. In Figure 3 the histogram of the players' solutions in a subnetwork of Wikipedia dataset with 1000 nodes is depicted.

For speeding up our simulation process, we use Lemma 2.3 whenever it can be applied.

3.1 Simulation Scenario

We consider three online social networks that have been used by Leskovec [15]: the trust network of the Epinions, the social network of the blog Slashdot, and the voting network of Wikipedia. While these datasets have hundreds of thousand nodes, for each one, we randomly extract an induced subnetwork that contains 1000 nodes. Moreover, these subnetworks are directed and we make them undirected by the procedure which is described in Table 1. We have two convincing reasons for this process of making undirected: (i) These datasets have very small fraction of reciprocated edges that have different signs(%0.0032 for Epinions, %0.0037 for Slashdot, and %0.0273 for Wikipeida), so the number of deleted edges is too small. (ii) In the balance theory (as opposed to status theory [15]) it is reasonable to expect that the reciprocated edges have same signs since the excessive edges should not have conflict with structural balance property of signed networks.

In addition to this real subnetworks, we use two randomly signed networks which are available online[1]. One is a $G_{n,p}$ and the other is a random power law graph, both with 1000 nodes. we start by these networks as initial signed networks. At each time unit, a node is chosen and computes her best-response, i.e., connects some edges to other nodes and chooses their sign. This process is repeated until all nodes apply their best-responses.

[1]You can find this datasets at : http://snap.stanford.edu/na09/

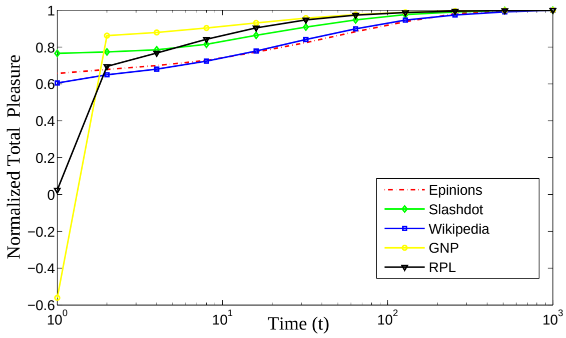

Figure 4: Normalized Total Pleasure of all nodes of the signed network in time units which are power of two. In each time step one node computed her best-response with QCQP maximization method and rounded its solution to values in $\{-1, 0, +1\}$. Each line represents temporal $\mathbb{P}$ for related real or random signed network.

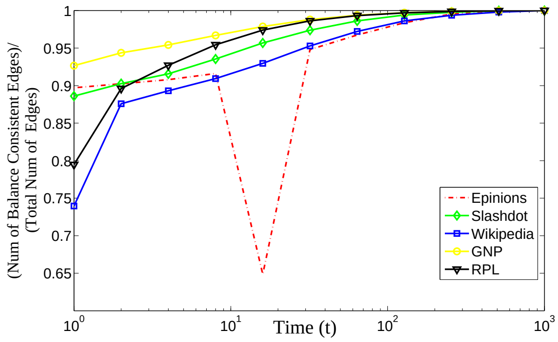

Figure 5: The fraction of edges that are consistent with global view of balance theory computed by Leskovec's method in [15]. After that each node played her best-response we enumerate the number of balance consistent edges with a maximization heuristic method. Each line represents temporal fraction of balance consistent edges for related real or random signed network.

Normalized Total Pleasure. We define *normalized total pleasure* ($\mathbb{P}$) as the pleasure of all nodes divided by total number of triangles in the underlying network:

$$\mathbb{P} = \frac{\sum_i p_i}{\sum_{i,j,k} \mid A_{ij} \mid \times \mid A_{jk} \mid \times \mid A_{ki} \mid} \tag{5}$$

In Figure 4, we calculate the normalized total pleasure in time units which are power of two. As we can see, independent of the initial value of $\mathbb{P}$, the normalized total pleasure of the network increases as time goes. One of the main observations of this experiment in random networks is that once the first player plays her best-response, many of the existing unbalanced triangles become balanced and, consequently, the normalized total pleasure of the underlying network substantially increases. Later on, this quantity increases at a decreasing rate. That is, whenever the first few changes in the structure of a random network are made based on the best-response model, the structure of random signed networks approaches to real-world signed networks. Another interesting note about Figure 4 is that in the real-world signed networks, unlike random networks, there is a high initial value for $\mathbb{P}$. The main reason for this difference is that real-world connections are formed according to structural balance theory which makes the pleasure received by each node in these networks high at the beginning.

Balanced Consistent Edges. One of the other properties of the signed networks, is the number of edges that are consistent with Cartwright-Harary theorem [4]. The Cartwright-Harary theorem suggests a global view of structural balance: the network can be divided into two mutually opposite sets of friends, i.e. every pair of nodes that are in the same set like each other, and every pair of nodes that are in the different sets are the enemies. In each time unit we calculate the number of edges that are consistent with this theorem by a maximization heuristic method that is also implemented and analysed by Leskovec [15]: we start by randomly partitioning the nodes into two sets. Then repeatedly pick a random node, and change the set it belongs to if that increases the number of positive edges with both ends in the same set, plus the number of negative edges with ends in opposite sets. Again we run the simulation and enumerate the number of balanced consistent edges in time units which are power of two. Figure 5 shows the temporal fraction of edges that are consistent with generalized balance theory. The existing spike in Epinions diagram is because of the heuristic basis of the method. As nodes change their relation based on best-response model, this quantity becomes large and after a few time units it gets sufficiently close to one.

Smartness Factor. One interesting way in analyzing nodes' behaviors is comparison between the current pleasure that the nodes have, and the new pleasure that they achieve if they play their best-response. Again, we randomly extract an induced subnetwork from each dataset that contains 1000 nodes. We randomly select 100 nodes in each subnetwork and calculate these two parameters: (i) $p_i(0)$ that is the pleasure of node i in its current state. (ii) $p_i(BR)$ that is the pleasure of node i if she uses her best-response to the remainder of the network. Our results show that the former has a wide variety in its value, but the latter has an almost constant value. Based on these two parameters a *smartness factor* is defined:

$$s_i = \frac{p_i(0)}{p_i(BR)} \tag{6}$$

This factor shows currently, how close is node i to her best-response. As shown in Figure 6, there is a big difference between these two values; this can be explained by the assumption that nodes have limited computational power to compute their best response.

KORIP Model. As it was mentioned earlier, Szell [18] shows that a vast majority of changes in the signed networks are due to the creation of new positive and negative links and not because switching of existing link from plus to minus or vice versa. On the other hand, it is a natural assumption that when two people are friends/enemies, they remain as friends/enemies for a long time. According to these two experimental and theoretical approaches, we introduce an

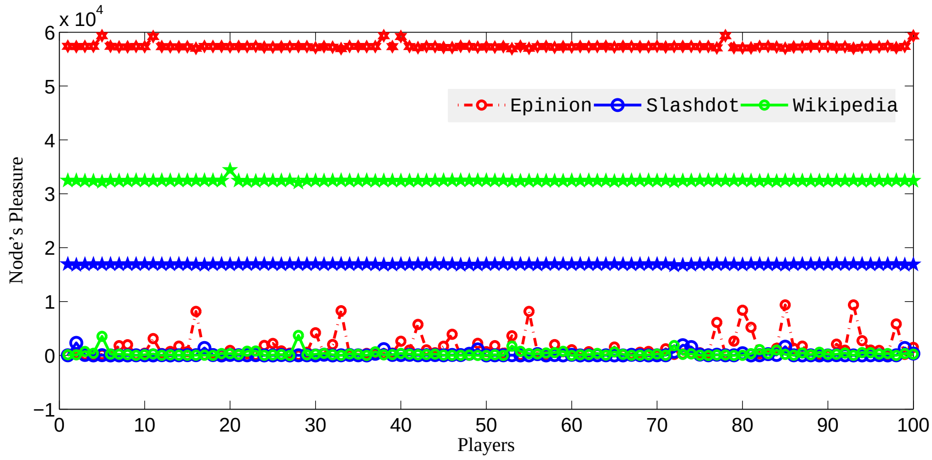

Figure 6: Nodes' pleasure, below lines show current pleasure and above lines show best-response pleasure

evolutionary model for signed social media that captures this property in a better way.

This model is similar to our best-response model except that when a node decides to play her best-response she Keeps her Old Relations but Improves her Pleasure (KORIP) by creating new connections that are suggested in the best-response. In other word, in this extended model when a node decides to play her best response, she considers two new restrictions: (i) she cannot change the sign of her old relations and (ii) she creates new links, based on her best response, only if these new links improve her pleasure in the network. The specification of the rest of the model remains unchanged.

We simulated the model in two $G_{n,p}$ and random power law graphs with 1000 nodes in which the difference between the number of positive and negative links in it was very small. By considering these graphs as initial signed networks, we let each node play her best response several times according to KORIP restrictions. Unlike the best-response model that reaches a fully balanced network after n times, the KORIP model stops in a an approximately balanced network. As depicted in Table 2, the characteristics of the above signed graphs gets close to the characteristic of the online social media sites after $2n$ times.

4. CONCLUSIONS

Human social networks are often a mixture of friendly and hostile relations that are usually modeled by signed networks. Structural balance theory is a basic framework to interpret the interactions between signs and their effects on the dynamics of the signed networks. This theory discusses every possible type of triadic relations that is formed among people in such networks. Triads that contain odd number of negative signs are said to be unbalanced because there are some pressure or stress on the people involved to change their sentiments. Studies on real-world signed networks (such as [18, 15]) show that these changes lead the network toward an approximately balanced state, where balanced triangles dominate unbalanced ones. We present a model of network formation games that capture the evolution of signed networks based on structural balance theory.

We investigate the theoretical aspects of the game. We prove that computing the best response for a player is NP-hard; we also obtain a good approximation for the best-response. We further prove that after one round the graph of the game will be complete and fully balanced. Also, we examine the best-response model on three online data sets and two types of random networks. We find that in the real-world signed networks, unlike random networks, there is a high initial value for total pleasure because of real-world connections formed according to structural balance theory. Furthermore, we conclude that in each signed network for new users there is an upper bound on the pleasure that they can achieve. Finally, we introduce a variant of the best-response model in which at each time step a user keeps her old relations and improves her pleasure by only creating new connections. We realized that this modified model can fit better in real signed networks modellings.

There are several directions for additional research. First, we aim to have more theoretical and experimental investigation on the KORIP model since its initial results show that it can be useful for modeling formation and evolution of the real social media networks. Second, we can pursue the signed network modeling based on Davis theory [6] in which a triangle with three negative sign is also considered as a balanced triangle. This relaxation is accepted and verified in various social media analysis and, therefore, has considerable motivation for further modeling. Finally, we suggest study of directed networks, where links among people need not be symmetric. In directed network we confront with status theory [15] which postulates that when person i makes a positive link to person j, then i is asserting that j has higher status.

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Table 2: Characteristic of initial and final signed graph according to KORIP model. T0, T1, T2 and T3 are columns which are used to show the number of triangles of types d, c, b and a of Figure 1 respectively.

	total pleasure	total triangles	balaced consistent edges	ubalaced consistent edges	T0	T1	T2	T3
initial GNP	109	334	8414	2526	41	119	85	103
KORIP GNP	142635186	144442865	945810	1960	226500	107963001	678228	35576025
initial RPL	1959	3218	24698	1947	182	1962	461	627
KORIP RPL	79249412	80559455	737350	1981	162684	59991528	492588	19912906

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